1. Coordinate Systems.

The usual equatorial coordinate system has the z-axis pointing to the North Celestial Pole (NCP), the x-and y-axes in the equatorial plane with the x-axis pointing toward the equinox. It is more convenient for present purposes to have the x-axis pointing to the NCP, and the negative z-axis pointing toward the equinox. Thus a unit line-of-sight (LOS) vector in this coordinate system toward an object with right ascension α and declination δ is

$$\mathbf{v_{eq}} = (s\delta, \quad c\delta s\alpha, \quad -c\delta c\alpha)' \tag{1.1}$$

The axes for a local level coordinate system ("altaz" or "NED" coordinate system) are chosen with the x-axis level and pointing north, the z-axis down. The unit LOS vector in this coordinate system toward an object with azimuth ψ and altitude θ is

$$\mathbf{v_{aa}} = (c\theta c\psi, \quad c\theta s\psi, \quad -s\theta)' \tag{1.2}$$

The final coordinate system has the x-axis aligned with the telescope, y and z in the field-of-view, with z down. If the telescope has the object centered in the FOV, then the unit LOS vector

$$\mathbf{v_t} = \mathbf{e} = (1, \quad 0, \quad 0)' \tag{1.3}$$

2. Transformations.

The transformation from equatorial to altaz consists of two rotations, sidereal time *t* about the x-axis, then latitude *l* about the -y-axis:

$$\mathbf{v_{aa}} = \mathbf{LSv_{eq}} \tag{2.1}$$

Written out, the two rotations are given by the 3×3 orthogonal matrices:

$$\mathbf{L} = \begin{pmatrix} c\ell & 0 & s\ell \\ 0 & 1 & 0 \\ -s\ell & 0 & c\ell \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & ct & st \\ 0 & -st & ct \end{pmatrix} \tag{2.2}$$

Similarly, the transformation from altaz to the telescope coordinates pointing at azimuth ψ and altitude θ also consists of two rotations, a rotation ψ about the z-axis, then θ about the resulting y-axis:

$$\mathbf{v_t} = \Theta\Psi\mathbf{v_{aa}} \tag{2.3}$$

with

$$\Theta = \begin{pmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{pmatrix}, \quad \Psi = \begin{pmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.4}$$

Combining (2.1) and (2.3),

$$\mathbf{v_t} = \Theta\Psi\mathbf{LSv_{eq}} \tag{2.5}$$

3. Differentials.

Differential changes of these matrices can be expressed in terms of the antisymmetric "unit rotators":

$$\mathbf{U_x} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \mathbf{U_y} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{U_z} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.1}$$

For example, taking the differential of the azimuth rotation matrix Ψ gives, in terms of a small azimuth angular change $\delta\psi$,

$$\delta\Psi = \delta\psi \begin{pmatrix} -s\psi & c\psi & 0 \\ -c\psi & -s\psi & 0 \\ 0 & 0 & 0 \end{pmatrix} = \delta\psi\mathbf{U_z}\Psi \tag{3.2}$$

4. Pointing Errors.

Errors $\delta\alpha$, $\delta\delta$ in right ascension and declination, respectively, produce the equatorial LOS error vector

$$\delta\mathbf{v_{eq}} = (s(\delta+\delta\delta), c(\delta+\delta\delta)s(\alpha+\delta\alpha), -c(\delta+\delta\delta)c(\alpha+\delta\alpha)) - \mathbf{v_{eq}} \tag{4.1}$$

With the rotation matrices for pointing at the catalog position of an object, the equatorial error vector $\delta\mathbf{v_{eq}}$ produces, from (2.5), the pointing error vector in telescope coordinates:

$$\delta\mathbf{v_t} = (\Theta\Psi\mathbf{LS})\delta\mathbf{v_{eq}} \tag{4.2}$$

On the other hand, rotation errors $\delta t, \delta\ell, \delta\psi, \delta\theta$ produce the telescope pointing error

$$\begin{aligned} \delta\mathbf{v_t} &= (\delta\Theta\Psi\mathbf{LS} + \Theta\delta\Psi\mathbf{LS} + \Theta\Psi\,\delta\mathbf{LS} + \Theta\Psi\mathbf{L}\,\delta\mathbf{S})\mathbf{v_{eq}} \\ &= (\delta\theta\mathbf{U_y}\Theta\Psi\mathbf{LS} + \delta\psi\Theta\mathbf{U_z}\Psi\mathbf{LS} - \delta\ell\Theta\Psi\mathbf{U_y}\mathbf{LS} + \delta t\Theta\Psi\mathbf{LU_x}\mathbf{S})\mathbf{v_{eq}} \end{aligned} \tag{4.3}$$

Using (1.3) and (2.5),

$$\delta\mathbf{v_t} = (\mathbf{U_y e})\delta\theta + (\Theta\mathbf{U_z}\Psi\mathbf{v_{aa}})\delta\psi - (\Theta\Psi\mathbf{U_y v_{aa}})\delta\ell + (\Theta\Psi\mathbf{LU_x Sv_{eq}})\delta t \tag{4.4}$$

To use this equation for evaluating the accuracy of altaz alignment, it is more convenient to use (2.1) in the last term,

$$\delta\mathbf{v_t} = (\mathbf{U_y e})\delta\theta + (\Theta\mathbf{U_z}\Psi\mathbf{v_{aa}})\delta\psi - (\Theta\Psi\mathbf{U_y v_{aa}})\delta\ell + (\Theta\Psi\mathbf{LU_x L' v_{aa}})\delta t \tag{4.5}$$

Set up the state vector $\mathbf{x} = (\delta\theta, \delta\psi, \delta\ell, \delta t)'$, and the observation vector formed from the last two elements of $\delta\mathbf{v_t}$, $\mathbf{z} = (\delta\mathbf{v_t}(2), \delta\mathbf{v_t}(3))'$, then the 2×4 observation matrix $\mathbf{H}$ such that $\mathbf{z} = \mathbf{Hx}$ is formed from the last two elements of each vector coefficient in (4.5). Let $\mathbf{z_1}, \mathbf{z_2}, \mathbf{H_1}, \mathbf{H_2}$ be the observation vectors and matrices for two alignment stars and

$$\mathbf{z_4} = \begin{bmatrix} \mathbf{z_1} \\ \mathbf{z_2} \end{bmatrix}, \quad \mathbf{H}_4 = \begin{bmatrix} \mathbf{H}_1 \\ \mathbf{H}_2 \end{bmatrix} \tag{4.6}$$

Assuming $\mathbf{H}_4$ is non-singular and the errors of the observations are uncorrelated with unit variance, the covariance matrix of the rotation errors is

$$E\{\mathbf{xx}'\} = E\left\{\mathbf{H_4}^{-1}\mathbf{z_4}\mathbf{z_4}'(\mathbf{H_4}')^{-1}\right\} = \left(\mathbf{H_4}'\mathbf{H_4}\right)^{-1} = \left(\mathbf{H_1}'\mathbf{H_1} + \mathbf{H_2}'\mathbf{H_2}\right)^{-1} = \mathbf{P} \tag{4.7}$$

The covariance matrix of the pointing errors resulting from this alignment is **HPH’** where **H** is the observation matrix at another star.